Entropy as a Measure of Uncertainty for PERT Network Completion Time Distributions and Critical Path Probabilities

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This study proposes that the entropy function provides a simple and useful tool for project managers to better cope with project uncertainty, and therefore better manage projects. The entropy function has a long and established history as a measure of uncertainty in information theory. If selected activities on PERT networks are modified by reducing their activity times by one or more time units, simulations on the original and modified networks can generate output on the entropies of completion time distributions and critical path probabilities. Modified networks can be ranked on a scale of decreasing entropy (or decreasing uncertainty) to determine which activities on the original network have the greatest impact on an overall reduction in project uncertainty. In this manner, these activities would be identified as worthy of additional resources in the real world to implement reductions in their activity times.

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I. INTRODUCTION

The objective of this study is to provide project manager practitioners with an additional tool for project control. More specifically, activities on a PERT simulation network modeling a project could be modified with additional resources, such as providing more productive machinery to the team assigned to an activity and/or increasing the size of the team, in order to reduce the time duration of completing the activity. With an appropriate measure for uncertainty, differences in the uncertainty can be measured among the completion time distributions generated by simulations of the original and modified networks, determining which modifications had the greater impact on uncertainty reduction and were deserving of the additional resources. Since critical path probability sets are a second source of uncertainty in PERT simulation, simulation output generating such sets provides an indicator of whether entropy reductions on the network may have impacted critical paths either more or less than completion times.

Achieving the objective requires two considerations: (1) completion time distributions and critical path probability sets must be measured with a high level of precision to assure that they reflect their true values, and (2) a suitable measure must be found that can rank modified completion time distributions based on their impact in reducing the uncertainty (or randomness or noise) of the project.

In reference to the first consideration, the classical PERT technique (Malcolm et al., 1959) is not suitable for this study because it fails to generate valid representations of completion time distributions. The technique is subject to “PERT bias,” which refers to distortions of the true completion time distribution that include a significant understatement of the mean (MacCrimmon and Ryavec, 1964). However, Monte Carlo techniques and PERT Network Simulation where developed to address the shortcomings of classical PERT (Van Slyke,
1966). PERT simulation can generate valid approximations of the completion time distribution where the precision of the approximation is subject to the number of simulated replications on the network. Following Van Slyke’s study in 1966, network simulation packages have evolved in the literature for over forty years. GERT (Graphical Evaluation and Review Technique) simulation was an enhancement to PERT which permitted probabilistic branching on AOA (activity on arc) networks (Pritsker and Happ, 1966; Moore and Clayton, 1976). PLANET (Project Length Analysis and Evaluation Technique) was a modification of GERT which accommodated criticality indices (Kennedy and Thrall, 1976). VERT (Venture Evaluation and Review Technique) shares many of the features of GERT and was independently developed for the military (Moeller and Digman, 1981). CAPERTSIM (Computer Assisted PERT SIMULATION) was developed primarily as a project management teaching tool and accommodated cost-time tradeoffs (Ameen, 1987). STARC was developed to allow a duration risk factor to be measured as a percentage over a time range for extended activities (Badiru, 1991). Other studies extended PERT simulation to stochastic forms of CPM (Herbert, 1980; Johnson and Schou, 1990). More recent studies based on PERT simulation include: the development of a set of project management tools which perform simulation on Crystal Ball spreadsheets (Meredith and Mantel, 2002); a Java based CPM simulation package referred to as SPSS (Stochastic Project Scheduling Simulation) developed for construction projects (Lee, 2005), an animation of a PERT network in Arena as a pedagogical tool (Cosgrove, 2006), and the development of Arena-based module groups which permit the quick construction of PERT simulation networks in a Microsoft Windows environment (Cosgrove, 2008). In the practitioner market, several commercial software products have been developed for network simulation. These tools not only employ simulation to address the shortcomings of classical PERT, but also piggyback on the popularity of Microsoft Project (http://office.microsoft.com/en-us/project/default.aspx) as add-on tools, providing extended features such as simulation, probabilistic and conditional branching, and extended graphics enhancements. Some of the most popular products include Deltek Risk+ (http://www.somos.com/files/ds_risk+.pdf), Palisade’s @RISK Professional for Project (http://www.palisade.com/riskproject/), and Oracle’s Primavera Risk Analysis (http://www.oracle.com/applications/primavera/primavera-risk-analysis-data-sheet.pdf).

The second consideration of the objective relates to finding a suitable measure for uncertainty. While the temptation is to associate risk and uncertainty as the same, two studies that focus on project uncertainty attempt to distinguish between project risk and project uncertainty (Chapman and Ward, 2000; Ward and Chapman, 2003). Both studies place a greater emphasis on developing concepts and processes to cope with uncertainty, rather than developing operational definitions to quantify uncertainty as a phenomenon that can be measured. Consequently, this study addresses the second consideration by proposing the entropy function as a measure of uncertainty that can be applied to completion time distributions and critical path probability sets which are typical outputs from PERT network simulation. Since its introduction in 1948 (Shannon, 1948), entropy has enjoyed a long and rich tradition for over 60 years as a measure of uncertainty in probabilistic information theory (Verdu and McLaughlin, 2000).

In the sections that follow, entropy is proposed as a measure of project uncertainty based on several key mathematical properties found in the literature on information theory. Applications of these properties are demonstrated with simulations on a PERT network employing Arena (Cosgrove, 2008; Kelton et al., 2010).

II. THE ENTROPY FUNCTION
Consider P as a probability set of order R with members \( p_r \) such that
\[
P = \{p_r | 0 \leq p_r \leq 1; \sum p_r = 1; r=1,2,...,R \}. \quad (1)
\]
The general form of the entropy function for any probability set P satisfying (1) is given by (Shannon, 1948)
\[
H(P) = - \sum_{p_r \in P} p_r \log_b p_r. \quad (2)
\]
There are no rigid restrictions on setting the base of the logarithm, which in this study will be taken at the base e (i.e., \( b=e \)) at the discretion of the author.

Integral versions of the entropy function exist for continuous distributions, but they are of no use in this study because the output from PERT simulation generates discrete probability sets for critical paths and discrete distributions for completion time.

Several key mathematical properties that follow support the use of the entropy function as a measure of uncertainty. These properties can be found in numerous books and articles on information theory and statistics (Reza, 1961; Jelinek, 1968; Hays and Winkler, 1975; Jones, 1979):

1. Min[H(P)] = 0, which follows if there exists a single value of \( r \) such that \( p_r = 1 \), where the \( r \)th state is deterministic.

2. The logarithm of zero is undefined, but Expression (2) is finite since
\[
\lim_{p_r \to 0} [p_r \log_b p_r] = 0. \quad (3)
\]

3. Max[H(P)] = \( \log R \) for any discrete distribution with R nonzero probabilities. This property follows intuition in that maximum entropy occurs when all states are equiprobable (i.e., \( p_r = 1/R \) for all \( r \) where \( p_r \neq 0 \)).

4. \( H(P) \) is a function of probabilities and is therefore distribution free. It is suitable for both nominal/categorical, ordinal, and metric data.

5. \( H(P) \) is a continuous measure that increases over the range from zero to \( \log R \).

These properties support the notion of entropy as a continuous and relative measure of uncertainty. In this manner, entropy can be employed to rank random variables on an ordinal scale based on the entropy of their underlying distributions. Note that significant differences in uncertainty between or among random variables are not a topic of this study.

Another characteristic of the above properties follows when distinguishing between variance and entropy for metric data. Consider the distributions in Figure 1.

**FIGURE 1. COMPARISONS OF ENTROPY/VARIANCE**

Note that the first and second distributions have the same entropy, but the variance of the first is significantly less than the second. This supports the notion from information theory that an observer’s knowledge about future events represented by these distributions is exactly the same regardless of the values of their respective variances. An equivalent statement is that the uncertainty about the realization of future events for both distributions is the same. Now consider the third distribution where the variance exceeds that of the other two but the entropy is less. The three distributions have in common a single state...
with a .50 probability, but differ in that the first two distributions have two states each at .25, and the third distribution has two states at .20 and .30. It is evident that the two equiprobable states of .25 leave the observer less certain about the future realization of these states than the situation represented in the third distribution, where the observer would conclude the third state at .20 is less likely to occur and the fourth state at .30 is more likely to occur. This added knowledge about the third and fourth states of Distribution 3 leads to its lower level of uncertainty (i.e., entropy). However, the analysis as described above, which compared absolute probabilities over a very short range among only three distributions, would be too burdensome for practitioners working with distributions generated by simulation having much larger ranges. Since entropy values eliminate the need to work directly with subsets of absolute probabilities, the ranking of distributions by their uncertainty is accomplished by simply applying Expression (2).

In the sections that follow, separate entropy functions based on Expression (2) are developed for completion time and critical path. Applications of entropy on simulation output are included to demonstrate that activities undergoing time reduction can be ranked for their impact on project uncertainty, with this ranking based on entropy measurements of completion time distributions and/or critical path probability sets.

III. ENTROPY FUNCTIONS FOR COMPLETION TIME AND CRITICAL PATH

Consider the random variable Y representing the total completion time of a PERT network with distribution function $P(Y)$ and variates $y_j$ such that $Y = \{ y_j | j=1,2,\ldots,J \}$. It follows from Expression (2) that the entropy function of $P(Y)$ is given by

$$H(Y) = - \sum_{y_j \in Y} p(y_j) \ln p(y_j).$$

Likewise, consider the random variable X for critical path with probability set $P(X)$ and variates $x_i$ such that $X = \{ x_i | i=1,2,3,\ldots,I \}$. The entropy function of $P(X)$ is given by

$$H(X) = - \sum_{x_i \in X} p(x_i) \ln p(x_i).$$

Expressions (4) and (5) provide measures of uncertainty for two key random variables which are typically generated in PERT simulations. Some straightforward applications of these expressions for practitioners are illustrated in the next section.

IV. APPLICATION

To assure a high level of precision in the simulation output discussed in this section, each of five simulation runs underwent 10,000 replications which far exceeds the recommended run size of 1,000 (Moder et al., 1983).

It is assumed that opportunities exist to reduce the durations of some activity times on a PERT project network with the intent of reducing the overall uncertainty of the project, particularly by reducing the uncertainty in the completion time distribution. Reductions in activity times assume that resources are available to reduce the duration of one or more selected activities on the network, perhaps by increasing the size of the team assigned to these activities. Activities subject to time reduction would require a reformulation of their activity time distributions followed by a simulation run with the reformulated distributions. As shown in this section, a comparison of completion time entropies from this simulation with the original simulation would lead to a determination of the extent that activity time reductions actually reduced the uncertainty of the project.

Figure 2 includes a PERT network with completion time distributions generated from five
simulation runs, with specified time units along the horizontal axes with variates $y_1=13$, $y_2=14$, $y_3=15, \ldots y_23=35$. With the exception of dummy activities, activity time distributions and their parameters are specified for all activities on the network and appear below the completion time distributions for each run. Probability sets for critical path appear in Table 1 with top, middle, and bottom paths corresponding to the variates $x_1$, $x_2$, and $x_3$, respectively. As a convenience to the reader, the time units on the horizontal axes for the completion time distributions are vertically aligned in Figure 2, and entropies for critical paths and completion time appear in both Figure 2 and Table 1.

It is assumed that the original simulation is represented by Run 1, and any activity time changes from Run 1 to other runs are shown in bold type under their respective completion time distributions. It is also assumed that the cost for activity time reductions and modifications are the same for all five runs. Note that this includes Run 5, which has modifications on four activities (C, D, E, and F) which combined is assumed to be the same cost for time modifications on each of the activities in Runs 2, 3, and 4 (i.e., on B, H, and C).

Assume that resources are available to reduce the most pessimistic times of activities B or H by one time unit, or replace C's
triangular distribution with a discrete distribution. Runs 2 and 3 represent the modifications on B and H and Run 4 the modification on C.

The output in Figure 2 shows that Runs 2 and 3 outperform Run 1 in all the measures shown [i.e, in average, variance, $H(X)$, and $H(Y)$]. Run 2 outperforms Run 3 on average, variance and $H(Y)$, but underperforms on $H(X)$. This indicates that the time reduction on B in Run 2 has a greater impact on reducing completion time uncertainty than critical path uncertainty, with the opposite occurring with Run 3 from the time reduction on H. Since project managers are more interested in reducing completion time uncertainty over critical path uncertainty, and since Run 2 has a better average and variance than Run 3, the best result for reducing project completion time uncertainty is to direct resources to time reduction on activity B.

Runs 4 and 5 generated multimodal distributions which led to their large variances when compared to Run 1. Since C, D, and E are always on the critical path, the critical path probabilities and $H(X)$ are the same for Runs 1, 4, and 5 (see Table 1). However, while Run 5 exhibits the largest variance, it also exhibits the lowest mean and the lowest completion time entropy of all the alternatives. It would likely be the preferred alternative among the five runs for most practitioners.

V. CONCLUSION

Traditional approaches to simulation output analysis from PERT networks have tended to focus more on completion time and critical paths than on uncertainty. This study argues that the entropy function offers an additional tool to practitioners to ask basic “what if” questions concerning the stochastic nature of projects. This suggests new options to explore tradeoffs with time and uncertainty, better recognizing that the network model, if properly constructed and validated, can be manipulated to reduce some of the random factors that make many projects so difficult to manage and control.

Extensions of this study include multivariate versions of the entropy function which include measures for the interaction between and among random variables (e.g., such as completion times, criticality indices, critical path probabilities, and variations in network structure) to measure information gain and redundancy in project network simulation models. Such measures, while commonly employed in information theory and in the study of communications, are ideal for stochastic project networks that go beyond PERT, such as GERT and VERT networks which are an enhancement of the basic PERT network by permitting probabilistic branching and probability sets for varying network structures.

VI. REFERENCES

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